Rational solutions of Knizhnik-Zamolodchikov system

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Abstract

We consider Knizhnik-Zamolodchikov system of linear differential equations. The coefficients of this system are rational functions. We prove that under some conditions the solution of KZ system is rational too. This assertion confirms partially the conjecture of Chervov-Talalaev.
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1 Main Theorem

1.We consider the differential system

$$\frac{dW}{dz} = \rho A(z)W, \quad z{\in}C, \tag{1.1}$$

where ρ is integer, $A(z)$ and $W(z)$ are $n{\times}n$ matrix functions. We suppose that $A(z)$ has the form

$$A(z) = \sum_{k=1}^{s} \frac{P_k}{z - z_k}, \tag{1.2}$$

where $z_k{\neq}z_\ell$ if $k{\neq}\ell$. In a neighborhood of z_k the matrix function $A(z)$ can be represented in the form

$$A(z) = \frac{a_{-1}}{z - z_k} + a_0 + a_1(z - z_k) + ..., \tag{1.3}$$

where a_k are $n{\times}n$ matrices. We investigate the case when z_k is either a regular point of $W(z)$ or a pole. Hence the following relation

$$W(z) = \sum_{p\geq m} b_p(z - z_k)^p, \quad b_m{\neq}0 \tag{1.4}$$

is true. Here b_p are $n{\times}n$ matrices. We note that m can be negative.
Proposition 1.1.(necessary condition, (see **[Sa06]**) *If the solution of system* (1.1) *has form* (1.4) *then* m *is an eigenvalue of* a_{-1}.
We denote by M the greatest integer eigenvalue of the matrix ρa_{-1}. Using relations ((1.3) and (1.4)) we obtain the assertion.
Proposition 1.2.(necessary and sufficient condition, (see **[Sa06]**) *If the matrix system*

$$[(q+1)I_n - a_{-1}]b_{q+1} = \sum_{j+\ell=q} \rho a_j b_\ell, \tag{1.5}$$

where $m{\le}q+1{\le}M$ *has a solution* $b_m, b_{m+1}, ..., b_M$ *and* $b_m{\ne}0$ *then system (1.1) has a solution of form (1.4).*
We introduce the matrices

$$P_k^- = I + P_k, \quad P_k^+ = I - P_k. \tag{1.6}$$

Theorem 1.1. *Let the following conditions be fulfilled*
1)

$$P_k^2 = I_n, \quad 1{\le}k{\le}s. \tag{1.7}$$

2)

$$[P_j P_k P_\ell + P_\ell P_k P_j]P_k^+ = 0, \quad j{\ne}k, \quad j{\ne}\ell, \quad k{\ne}\ell. \tag{1.8}$$

3)

$$(P_j P_k P_j + P_j)P_k^+ = P_k^+, \quad j{\ne}k. \tag{1.9}$$

4) The matrices $P_j \quad (1{\le}j{\le}s)$ are symmetric.
If $\rho = \pm 1$ *then system* (1.1), (1.2) *has a rational fundamental solution .*
Proof. We shall use the notations (1.2) and (1.3). Then in neighborhood of z_k we have

$$a_{-1} = P_k, \quad a_r = (-1)^r \sum_{j\ne k} \frac{P_j}{(z_k - z_j)^{r+1}}, \quad r{\ge}0. \tag{1.10}$$

Using relation (1.5) and equality

$$P_k^- P_k^+ = 0 \tag{1.11}$$

we have

$$b_{-1} = P_k^+, \quad b_0 = -\sum_{j\ne k} P_k \frac{P_j}{(z_k - z_j)} b_{-1}. \tag{1.12}$$

Formulas (1.5) and (1.12) imply that

$$P_k^{+}b_1=-[\sum_{j\neq k}\frac{P_j}{z_k-z_j}\sum_{\ell\neq k}P_k\frac{P_\ell}{z_k-z_\ell}+\sum_{j\neq k}\frac{P_j}{(z_k-z_j)^2}]b_{-1}. \tag{1.13}$$

Due to conditions (1.8) and (1.9) we can write (1.12) in the form

$$P_k^{+}b_1=-\sum_{j\neq k}\frac{1}{(z_k-z_j)^2}P_k^{+}. \tag{1.14}$$

Equation (1.14) has the following solution

$$b_1=-\beta_k I_n, \quad when \quad \beta_k=\sum_{j\neq k}\frac{1}{(z_k-z_j)^2}\neq 0 \tag{1.15}$$

and

$$b_1=P_k^{-}, \quad when \quad \beta_k=0. \tag{1.16}$$

Together with system (1.1) we consider the differential system

$$\frac{dY}{dz}=-Y(z)A(z), \quad z\in C, \tag{1.17}$$

where $A(z)$ is defined by (1.2).The strong regular solution of system (1.17) has the form (see **[Sa06]**):

$$Y(z)=\sum_{p\geq m}c_p(z-z_k)^p, \quad b_m\neq 0. \tag{1.18}$$

The following relations

$$c_{q+1}[(q+1)I_n+a_{-1}]=-\sum_{j+\ell=q}c_\ell a_j, \tag{1.19}$$

are true (see**[Sao6]**). Here $j\geq 0, \quad \ell\geq -1$. Then we have

$$c_{-1}=P_k^{-}, \quad c_0=-\sum_{j\neq k}P_k^{-}\frac{P_j}{(z_k-z_j)}P_k. \tag{1.20}$$

From relations (1.19) and (1.20) we deduce that

$$c_1P_k^{-}=P_k^{-}[\sum_{j\neq k}\frac{P_j}{z_k-z_j}\sum_{\ell\neq k}P_k\frac{P_\ell}{z_k-z_\ell}+\sum_{j\neq k}\frac{P_j}{(z_k-z_j)^2}]. \tag{1.21}$$

According condition (1.10) the matrix P_j coincides with the transposed matrix P_j^τ, i.e.

$$P_j = P_j^\tau, \quad 1 \le j \le s. \tag{1.22}$$

By relations (1.6),(1.7), (1.22) and the equalities

$$P_k^+ + P_k^- = 2I_n, \quad [P_k^+]^2 + [P_k^-]^2 = 4I_n \tag{1.23}$$

we obtain that

$$P_k^-[P_jP_kP_\ell + P_\ell P_kP_j] = [P_jP_kP_\ell + P_\ell P_kP_j]P_k^-, \quad j \neq k, \quad j \neq \ell, \quad k \neq \ell. \tag{1.24}$$

$$P_k^-(P_jP_kP_j + P_j) = (P_jP_kP_j + P_j)P_k^-, \quad j \neq k \tag{1.25}$$

Hence equation (1.21) has the solution

$$c_1 = \sum_{j \neq k} \frac{P_j}{z_k - z_j} \sum_{\ell \neq k} P_k \frac{P_\ell}{z_k - z_\ell} + \sum_{j \neq k} \frac{P_j}{(z_k - z_j)^2}, \quad when \quad \beta_k \neq 0 \tag{1.26}$$

and

$$c_1 = \sum_{j \neq k} \frac{P_j}{z_k - z_j} \sum_{\ell \neq k} P_k \frac{P_\ell}{z_k - z_\ell} + \sum_{j \neq k} \frac{P_j}{(z_k - z_j)^2} + P_k^+, \quad when \quad \beta_k = 0. \tag{1.27}$$

It follows from (1.1) and (1.17) that

$$\frac{d}{dz}[W(z)Y(z)] = 0. \tag{1.28}$$

Using (1.4) and (1.18) we obtain

$$W(z)Y(z) = b_0c_0 + b_{-1}c_1 + b_1c_{-1}. \tag{1.29}$$

Relations (1.12) ,(1.15),(1.16), and (1.20),(1.25),(1.26) imply that

$$b_0c_0 = 0, \quad b_{-1}c_1 + b_1c_{-1} = 2\beta_kI_n \quad when \quad \beta_k \neq 0, \tag{1.30}$$

and

$$b_{-1}c_1 + b_1c_{-1} = 4I_n \quad when \quad \beta_k = 0, \tag{1.31}$$

Hence we have

$$\det W(z)Y(z)\neq 0. \tag{1.32}$$

In view of (1.32) the constructed solutions $W(z)$ and $Y(z)$ of systems (1.1) and (1.17) respectively are fundamental. It follows from (1.2) that the point $z=\infty$ is the singular point of the first kind (see **CL55, Ch.4**).As in our case the fundamental solutions $W(z)$ and $Y(z)$ are one-valued, then the following representations

$$W(z)=\sum_{j=-\infty}^{m_1} g_j z^j, \quad m_1<\infty, \quad |z|>R, \tag{1.33}$$

$$Y(z)=\sum_{j=-\infty}^{m_2} h_j z^j, \quad m_2<\infty, \quad |z|>R \tag{1.34}$$

are true (see **CL55, Ch.4**). Here $g_j, \quad h_j$ are $n\times n$ matrices.Thus all the the points $z_k \quad (1\leq k\leq s)$ and $z=\infty$ are strong regular. Hence $W(z)$ and $Y(z)$ are rational matrix functions.We note that $Y^{\tau}(z)$ is the fundamental solution of system (1.1), when $\rho=-1$. The theorem is proved.

Remark 1.1. When $s=1$ conditions 1) and 2) of Theorem 1.1 must be omitted. When s=1 condition 2) of Theorem 1.1 must be omitted.

Remark 1.2. Let the matrices P_j be symmetric. If system (1.1),(1.2) has a fundamental rational solution $W(z)$, when $\rho=k$, then this system has the fundamental rational solution $Y(z)=[W^{-1}(z)]^{\tau}$, when $\rho=-k$.

Corollary 1.1. *Let conditions of Theorem 1.1 be fulfilled.Then the matrix functions $W(z)$ and $Y(z)$ can be written in the forms*

$$W(z)=\sum_{k=1}^{s}\frac{L_k}{z-z_k}+Q_1(z), \tag{1.35}$$

$$Y(z)=\sum_{k=1}^{s}\frac{M_k}{z-z_k}+Q_2(z), \tag{1.36}$$

where L_k and M_k are $n\times n$ matrices, $Q_1(z)$ and $Q_2(z)$ are $n\times n$ matrix polynomials.

Further we use the relation

$$A(z)=\frac{T}{z}[1+o(1)], \quad z\to\infty, \tag{1.37}$$

where

$$T = \sum_{k=1}^{s} P_k. \tag{1.38}$$

Relation (1.37) and the strong regularity of the point $z = \infty$ imply the following assertion.

Corollary 1.2. *All eigenvalues of the matrix T are integer.*

We denote by m_T the smallest eigenvalue and by M_T the greatest eigenvalue of T. Changing the variable $z = \frac{1}{u}$ in system (1.1) we obtain the following results.

Corollary 1.3. *Let matrix polynomials $Q_1(z)$ and $Q_2(z)$ be defined by relations* (1.35) *and* (1.36).

1. *If $M_T \geq 0$ then* $\deg Q_1(z) = M_T$.
2. *If $M_T < 0$ then* $Q_1(z) = 0$.
3. *If $m_T \leq 0$ then* $\deg Q_2(z) = -m_T$.
4. *If $m_T > 0$ then* $Q_2(z) = 0$.

2 Representation of the symmetric group S_n

Let S_n be the symmetric group. We consider the natural representation of S_n. By $(i; j)$ we denote the permutation which transposes i and j and preserves all the rest. The $n \times n$ matrix which corresponds to $(i; j)$ is denoted by

$$P(i, j) = [p_{k,\ell}(i, j)], \quad (i \neq j). \tag{2.1}$$

The elements $p_{k,\ell}(i, j)$ are equal to zero except the following cases

$$p_{k,\ell}(i, j) = 1, \quad (k = i, \ell = j); \quad p_{k,\ell}(i, j) = 1, \quad (k = j, \ell = i), \tag{2.2}$$

$$p_{k,k}(i, j) = 1, \quad (k \neq i, k \neq j). \tag{2.3}$$

Now we introduce the matrices

$$P_k = P(1, k + 1), \quad 1 \leq k \leq n - 1. \tag{2.4}$$

Proposition 2.1. *The matrices P_k, $(1 \leq k \leq n - 1)$ satisfy the conditions 1)-4) of Theorem 1.1.*

Proof. It follows from relations (2.1) - (2.4) that conditions 1) and 4) are fulfilled. By direct calculation we see that

$$[P_1P_2P_1 + P_1]P_2^+ = P_2^+, \quad n = 3, \tag{2.5}$$

$$[P_1P_2P_3 + P_3P_2P_1]P_2^+ = 0, \quad n = 4. \tag{2.6}$$

Hence conditions 2) and 3) are fulfilled for all j, k, ℓ and n.The proposition is proved.

Corollary 2.1. *The system* (1.1), (1.2) *has a rational fundamental solution, when* $\rho = \pm 1, \quad P_k = P(1, k+1)$.

Corollary 2.1.confirms the conjecture of A.Chervov and D.Talalaev (see[**CT06**] for the case $\rho = \pm 1$.

Remark 2.1. The natural representation of S_n is the sum of the 1-reperesentation and an irreducible representation (see [Bu65, p. 106]).

We introduce the $1 \times (n-1)$ vector $e = [1, 1, ..., 1]$ and $n \times n$ matrix

$$T_1 = \begin{bmatrix} 2-n & e \\ e^\tau & 0 \end{bmatrix}. \tag{2.7}$$

Using relations (1.38) and (2.1)-(2.4) we deduce that

$$T = (n-2)I_n + T_1. \tag{2.8}$$

The eigenvalues of T are defined by the equalities

$$\lambda_1 = n-1, \quad \lambda_2 = n-1, \quad \lambda_3 = -1. \tag{2.9}$$

Hence we have $m_T = -1, \quad M_T = n-1$. It follows from Corollary 1.3 the statement.

Proposition 2.2. *Let the matrices P_k are defined by relations* $(2.1)-(2.4)$. *Then the equalities*

$$\deg Q_1(z) = n-1, \quad \deg Q_2(z) = -1 \tag{2.10}$$

are true.

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